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2001 J. Phys. A: Math. Gen. 34 305

(http://iopscience.iop.org/0305-4470/34/2/307)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 305–316 www.iop.org/Journals/ja PII: S0305-4470(01)14712-8

Revisitation of the localized excitations of the (2 + 1)-dimensional KdV equation

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Received 12 June 2000, in final form 2 October 2000

Abstract

In the previous paper (Lou S-y 1995 *J. Phys. A: Math. Gen.* **28** 7227), a generalized dromion structure was revealed for the (2 + 1)-dimensional KdV equation, which was first derived by Boiti *et al* (Boiti M, Leon J J P, Manna M and Pempinelli F 1986 *Inverse Problems* **2** 271) using the idea of the weak Lax pair. In this paper, using a Bäcklund transformation and the variable separation approach, we find there exist much more abundant localized structures for the (2 + 1)-dimensional KdV equation. The abundance of the localized structures of the model is introduced by the entrance of an arbitrary function of the seed solution. Some special types of dromion solution, lumps, breathers, instantons and the ring type of soliton, are discussed by selecting the arbitrary functions appropriately. The dromion solutions can be driven by sets of straight-line and curved-line ghost solitons. The dromion solutions may be located not only at the cross points of the lines but also at the closed points of the curves. The breathers may breathe both in amplitude and in shape.

PACS numbers: 0230, 0220, 0540

AMS classification scheme numbers: 17B66, 17B68

1. Introduction

(1 + 1)-dimensional solitons and solitary wave solutions have been studied quite well both in the theoretical aspect and in the experimental aspect [3]. In (2 + 1) dimensions, some significant integrable models such as the Kadomtsev–Petviashvili (KP) equation [4], the Davey–Stewartson (DS) equation [5], the Nizhnik–Novikov–Vesselov (NNV) equation [6], the (2 + 1)-dimensional KdV equation (also named the asymmetric NNV (ANNV) equation, or BLMP (Boiti–Leon–Manna–Pempinelli) equation) [2] and the (2 + 1)-dimensional sine–Gordon (2DsG) [7] have also been established in nonlinear physics. Some special types of localized solution, dromions, which were first introduced in [8], for these (2 + 1)-dimensional

integrable models have also been obtained by means of some different approaches. Usually, the dromion solutions one finds are driven by straight-line solitons and the dromions are located at the cross points of the straight lines. For instance, for the DS and the NNV equations the dromion solutions can be obtained from two perpendicular line ghost solitons [9, 10], while for the KP equation the dromion solution can be driven by two non-perpendicular line solitons [11]. In [1, 12], one of the present authors (Lou) pointed out that the dromions may also be driven by curved-line solitons. From the symmetry study of the (2 + 1)-dimensional integrable models we know that there exist much more abundant symmetry structures than in (1 + 1) dimensions [13]. This fact hints to us that the localized solutions of the (2 + 1)-dimensional integrable models may have quite rich structures that have not yet been revealed. Using the classical Lie symmetry approach to the 2DsG equation, we find that the soliton solutions of 2DsG have much more abundant structures [14].

In this paper, we are interested to reveal the more abundant dromion structures for the (2 + 1)-dimensional KdV equation equation,

$$u_t + u_{xxx} - 3v_x u - 3vu_x = 0 \tag{1}$$

$$u_x = v_y \tag{2}$$

which was first derived by Boiti *et al* [2] using the idea of the weak Lax pair. The equation system (1) and (2) can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation [15]. In [15], we point out that (1) and (2) is an asymmetric part of the NNV equation. In the remainder of this paper, we call (1) and (2) the ANNV equation for simplicity.

To find some exact physically significant coherent soliton solutions (which are localized in all directions) in 2 + 1 dimensions is much more difficult than in 1 + 1 dimensions. There are a wealth of methods for finding special solutions of a nonlinear partial differential equation (PDE). Some of the most important methods are the inverse scattering transformation (IST) [16], the bilinear method [17], symmetry reductions [18], Bäcklund and Darboux transformations [19] and so on. In comparison with the linear case, it is known that IST is an extension of the Fourier transformation in the nonlinear case. In addition to the Fourier transformation, there is another powerful tool called the variable separation method in the linear case. Recently, two kinds of 'variable separating' procedure have been established. The first method is called the 'formal variable separation approach' (FVSA) [20], or equivalently the symmetry constraints or nonlinearization of the Lax pairs [21]. The independent variables of a reduced field in FVSA have not totally been separated though the reduced field satisfies some lower-dimensional equations. The second type of variable separation method had been established only for the DS equation and the asymmetric DS equation [22]. For the DS equation, by solving its bilinear form and introducing a prior ansatz, some special types of exact solution of the (2+1)-dimensional DS equation can be obtained from two (1+1)-dimensional variable separated fields [22].

In section 2 of this paper, we extend the second type of variable separation approach to find some special solutions of the ANNV equation. Two special linear variable separated equations can be found by means of a prior ansatz related to a special Bäcklund transformation. After solving the linear equations, a general solution of the ANNV equation with one arbitrary function of two independent variables $\{x, t\}$ and four other arbitrary functions of single independent variable with respect to y and t can be found. Some special types of localized structure are discussed in section 3 by selecting the arbitrary functions appropriately. Section 4 is a short summary and discussion.

2. Variable separation procedure for the ANNV equation

To use the variable separation approach, we take the following Bäcklund transformation:

$$u = -2(\ln f)_{xy} + u_0 \qquad v = -2(\ln f)_{xx} + v_0 \tag{3}$$

where $\{u_0, v_0\}$ is an arbitrary known seed solution of the ANNV equation.

Substituting (3) into (1) and integrating once with respect to x leads to a bilinear form

$$[D_t D_y + D_x^3 D_y - 3v_0 D_x D_y - 3u_0 D_x^2 + c(y,t)] f \cdot f = 0 \tag{4}$$

where $c(y,t)$ is an arbitrary integrating function and the operators D_t, D_x, D_y are defined as

$$D_x^m D_y^n D_t^k f \cdot f = \lim_{x'=x, y'=y, t'=t} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k \times f(x,y,t) \cdot f(x',y',t')$$

which were introduced first by Hirota [23]. Equation (2) is satisfied identically under the transformation (3).

To find some exact solutions of (4) via the variable separation approach, similar to the case in the DS equation [22], we looking for the solution of (4) in the form

$$f = 1 + a_1 p(x,t) + a_2 q(y,t) + A p(x,t) q(y,t) \tag{5}$$

where a_1, a_2 and A are arbitrary constants and $p \equiv p(x,t)$ and $q \equiv q(y,t)$ are functions of $\{x,t\}$ and $\{y,t\}$ respectively. It is clear that the variables x and y have now been separated totally to the functions p and q respectively.

Substituting the ansatz (5) into (4), we have

$$\begin{aligned}(A - a_1 a_2)(p_t + a p_{xxx} - 3v_0 p_x) - (a_2 + Ap)[(a_2 + Ap) - f q_y^{-1} \partial_y] q_t \\ +3(qA + a_1) q_y^{-1}((a_1 + Aq) p_x^2 - f p_{xx}) u_0 + c(y,t) f^2 q_y^{-1} = 0.\end{aligned} \tag{6}$$

Because p is y independent and q is x independent, equation (6) can be separated into two equations

$$p_t = 3v_0 p_x - p_{xxx} - (a_1 a_2 - A)(c_1 p^2 - c_3 p + c_2) \tag{7}$$

$$q_t = c_1(1 + a_2 q)^2 + c_2(a_1 + Aq)^2 + c_3(1 + a_2 q)(a_1 + Aq) \tag{8}$$

if we restrict the seed solution $\{u_0, v_0\}$ we have also a variable separation form

$$u_0 = 0 \qquad v_0 = v_0(x,t) \tag{9}$$

and the function $c(y,t)$ is fixed as

$$c(y,t) = -2(a_2^2 c_1 + a_2 A c_3 + A^2 c_2) q_y \tag{10}$$

where $v_0 \equiv v_0(x,t)$ is an arbitrary function of x and t, and $c_i \equiv c_i(t)$, $i = 1, 2, 3$ are arbitrary functions of t introduced by the variable separation procedure.

To give the general solutions of (7) and (8) for any fixed v_0 is still very difficult. Fortunately, because v_0 is an arbitrary function of x and t, we can treat the problem alternatively by considering p to be an arbitrary function of x and t and fixing the function v_0 by (7). The result reads

$$v_0 = (3p_x)^{-1}[p_t + p_{xxx} + (a_1 a_2 - A)(c_1 p^2 - c_3 p + c_2)]. \tag{11}$$

The general solution of (8) can also be obtained thanks to the fact that c_1, c_2 and c_3 are arbitrary functions. For the special case,

$$a_2^2 c_1 + a_2 A c_3 + A^2 c_2 = 0 \tag{12}$$

equation (8) is a linear equation only. The general solution in this special case can be written as

$$q = \exp\left(\int^t C_1(t')\,\mathrm{d}t'\right)\left[\int^t C_0(t')\exp\left(-\int^{t'} C_1(t'')\,\mathrm{d}t''\right)\mathrm{d}t' + F_1(y)\right] \tag{13}$$

where

$$C_1(t) = (a_1a_2 - A)a_2^{-1}(a_2c_3 + 2c_2A) \tag{14}$$

$$C_0(t) = (a_1a_2 - A)a_2^{-2}((a_2a_1 + A)c_2 + a_2c_3) \tag{15}$$

and $F_1(y)$ is an arbitrary function of y. For the general case,

$$a_2^2c_1 + a_2Ac_3 + A^2c_2 \neq 0 \tag{16}$$

we may rewrite c_1, c_2 and c_3 as

$$c_1 = \frac{A^2A_{2t}}{(a_1a_2 - A)^2} - \frac{A(a_1 + AA_2)A_{1t}}{(a_1a_2 - A)^2A_1} - \frac{(a_1 + AA_2)^2A_{3t}}{(a_1a_2 - A)^2A_1} \tag{17}$$

$$c_2 = \frac{a_2^2A_{2t}}{(a_1a_2 - A)^2} - \frac{a_2(1 + a_2A_2)A_{1t}}{(a_1a_2 - A)^2A_1} - \frac{(1 + a_2A_2)^2A_{3t}}{(a_1a_2 - A)^2A_1} \tag{18}$$

$$c_3 = \frac{(A + a_1a_2 + 2a_2AA_2)A_{1t}}{(a_1a_2 - A)^2A_1} - \frac{2a_2AA_{2t}}{(a_1a_2 - A)^2} + 2\frac{(1 + a_2A_2)(a_1 + AA_2)A_{3t}}{(a_1a_2 - A)^2A_1} \tag{19}$$

with $A_1 \equiv A_1(t)$, $A_2 \equiv A_2(t)$ and $A_3 \equiv A_3(t)$ being arbitrary functions of t. Using the relations (17)–(19), (8) becomes

$$q_t = \frac{-1}{A_1}[A_{3t}q^2 - (A_{1t} + 2A_2A_{3t})q - A_1A_{2t} + A_2A_{1t} + A_2^2A_{3t}]. \tag{20}$$

It is quite straightforward to verify that the general solution of (20) has the form

$$q = \frac{A_1}{A_3 + F_2(y)} + A_2 \tag{21}$$

where $F_2 \equiv F_2(y)$ is an arbitrary function of y.

Finally, substituting (5) into (3) we find that the ANNV equation possesses an exact solution

$$u = \frac{2q_yp_x(a_2a_1 - A)}{(1 + a_1p + a_2q + Apq)^2} \tag{22}$$

$$v = \frac{2(a_1 + Aq)^2p_x^2}{(1 + a_1p + a_2q + Apq)^2} - \frac{2(a_1 + Aq)p_{xx}}{(1 + a_1p + a_2q + Apq)} + v_0 \tag{23}$$

where p is an arbitrary function, q is given by (21) and v_0 is determined by (11) with (17)–(19).

3. Some special localized solutions

In [1,10], it was pointed out that the ANNV equation possesses some special types of coherent structure for the physical field u *rather than the potential* v. From the expression (22), we know that the ANNV equation has much more abundant coherent structure than those known in the literature thanks to the arbitrariness of the functions of p and c_i.

Generally, for arbitrary p and q with the boundary conditions

$$p|_{x\to-\infty} \to C_1 \qquad p|_{x\to+\infty} \to C_2 \qquad q|_{y\to-\infty} \to C_3 \qquad q|_{y\to+\infty} \to C_4 \tag{24}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants which may be infinities, (24) is a coherent soliton solution localized in all directions. Here are some interesting special examples.

3.1. Multi-dromion solutions driven by straight-line ghost solitons

Usually, the multi-dromion solutions which are localized in all directions are driven by multiple straight-line solitons with some suitable dispersion relations [10], and the dromions are located at the cross points of the straight lines. If we restrict the functions p and q as

$$p = \sum_{i=1}^{N} \exp(k_i x + \omega_i t + x_{0i}) \equiv \sum_{i=1}^{N} \exp(\xi_i) \tag{25}$$

$$q = \sum_{i=1}^{M} \exp(K_i y + y_{0i}) \sum_{j=1}^{J} \exp(\Omega_j t) \tag{26}$$

where x_{0i}, y_{0i}, k_i, ω_i, K_i and Ω_i are arbitrary constants and M, N and J are arbitrary positive integers, then we have the first type of special multi-dromion solution. The selection (26) corresponds to

$$A_3(t) = 0 \qquad F_2^{-1} = \sum_{i=1}^{M} \exp(K_i y + y_{0i}) \qquad A_1 = \sum_{j=1}^{J} \exp(\Omega_j t) \qquad A_2 = 0. \tag{27}$$

There is no dispersion relation among k_i, ω_i, K_i and Ω_i. The known dromion solutions given in the literature like [10] are just the case when one introduces the dispersion relations, $\omega_i = -k_i^3$ and fixes Ω_i as zero. From (7), we know that the dispersion relation, $\omega_i = -k_i^3$, has to be introduced if one fixes v_0 and c_i, $i = 1, 2, 3$ as $v_0 = c_i = 0$.

3.2. Multi-dromion solutions driven by curved-line ghost solitons

In [1], we pointed out that, for the ANNV equation, the dromion solutions can be driven not only by straight-line solitons but also by curved-line solitons. Actually, (22) can be rewritten as

$$u = \frac{Q_y P_x (a_2 a_1 - A)}{[\sqrt{A}\cosh(P + Q + \ln\sqrt{A}) + \sqrt{a_1 a_2}\cosh(P - Q + \ln\sqrt{a_1} - \ln\sqrt{a_2})]^2} \tag{28}$$

where Q and P are related to p and q by $p = \exp(2P)$, $q = \exp(2Q)$. So the general multi-dromion solutions of the ANNV expressed by (22) (or equivalently (28)) are driven by two sets of straight-line solitons and some curved-line solitons. If the first set of straight-line solitons appears in the factor Q_y, say, one can take

$$Q_y = \sum_{i=1}^{N} Q_i (y - y_{i0}) \tag{29}$$

where $Q_i = Q_i(y - y_{i0})$ denotes a straight-line soliton which is finite at the line $y = y_{i0}$ and decays rapidly away from the line. The second set of straight-line solitons appears in the factor P_x. For instance, any multi-soliton solutions of any 1 + 1-dimensional integrable models can be chosen as P_x. Of course P_x can also be taken as in the similar form of (29). Finally, the curved-line solitons are determined by the factors $\sqrt{A}\cosh(P + Q + \ln\sqrt{A})$ and $\sqrt{a_1 a_2}\cosh(P - Q + \ln\sqrt{a_1} - \ln\sqrt{a_2})$ of (28) and the curves are determined by

$$P + Q + \ln\sqrt{A} = \min(P + Q + \ln\sqrt{A}) \tag{30}$$

$$P - Q + \ln\sqrt{a_1} - \ln\sqrt{a_2} = \min(P - Q + \ln\sqrt{a_1} - \ln\sqrt{a_2}) \tag{31}$$

while the number of curved-line solitons is determined by the branches of the equations (30) and (31). The dromions are located at the cross points and/or the closed points of the straight and curved lines.

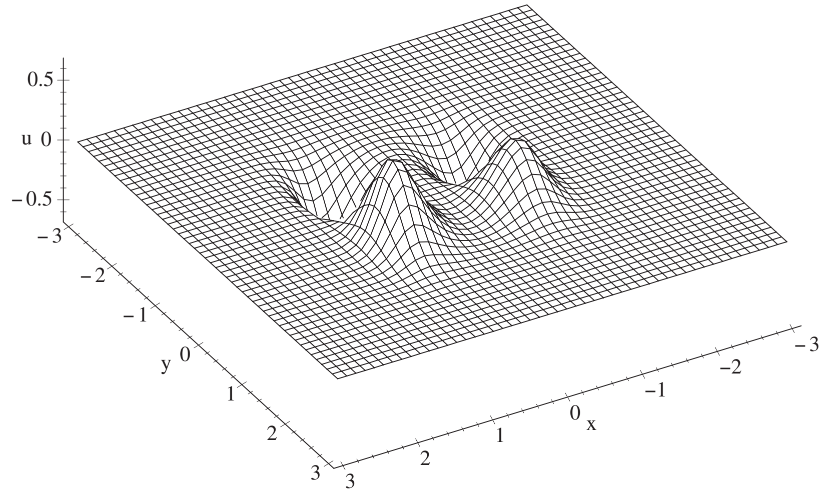


Figure 1. Static dromion solution driven by two curved-line solitons located at $x^3+y^2+\frac{1}{2}\ln 10=0$ and $x^3-y^2-\frac{1}{2}\ln 2=0$, which corresponds to $P=x^3$, $Q=y^3$, $A=a_1=10, a_2=20$.

Figure 1 is a plot of a static dromion solution by taking simply $Q=y^2$, $P=x^3$, $A=a_1=10, a_2=20$. The dromion shown by figure 1 is driven by two curved-line solitons and the curved lines have the forms

$$x^3+y^2+\tfrac{1}{2}\ln 10=0 \tag{32}$$

$$x^3-y^2-\tfrac{1}{2}\ln 2=0. \tag{33}$$

It should be pointed out there is no cross point of the curves (32) and (33) at all. Figure 1 shows us that a dromion can exist not only at the cross point of the lines but also at the point between the closed points of the curved and straight lines!

3.3. Multi-lump solutions

It is also known that in high dimensions, like the KP equation, a special type of localized structure (called lump solutions) may also be formed by rational functions. The situation occurs also in the ANNV case. Figure 2 shows a multi-lump structure of the ANNV equation, where p and q in (22) are simply fixed as $p=x^2$ and $q=y^2$ with the parameters $a_1=A=1$ and $a_2=2$.

3.4. Oscillating dromion solutions

If some periodic functions in space variables are included in the functions p and q, we may obtain some types of dromion solution with oscillating tails in certain directions. For instance, if we take

$$p=\exp(x(\cos x+4/3)) \qquad q=\exp(y) \tag{34}$$

then we can obtain a dromion solution oscillating in the x direction.

Figure 3 is a plot of the oscillating dromion structure with the condition (34) and $a_1=a_2/2=A=1$.

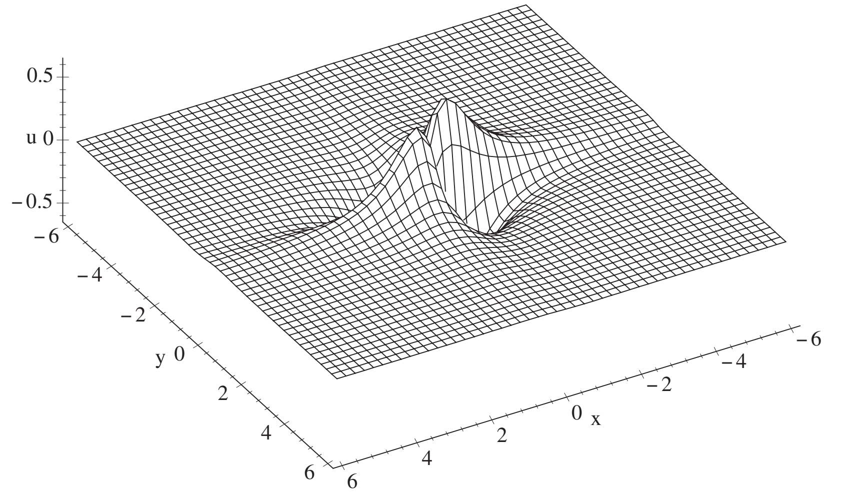


Figure 2. A multi-lump structure of the ANNV equation for $p = x^2, q = y^2$ and $a_1 = a_2/2 = A = 1$.

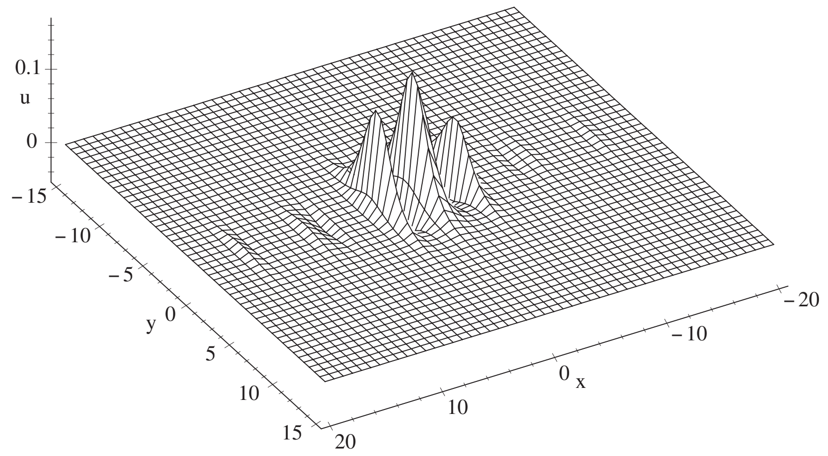


Figure 3. An oscillating dromion solution with the condition (34) and $a_1 = a_2/2 = A = 1$.

3.5. Ring soliton solutions

In high dimensions, in addition to the point-like localized coherent excitations, there may be some other types of physically significant localized excitation. For instance, in (2 + 1)-dimensional cases, there may be some types of ring soliton solution which are not equal to zero identically at some closed curves and decay exponentially away from the closed curves. For the ANNV equation, some special types of ring soliton solution can also be found. For instance, if the functions P and Q in (28) are selected as

$$P_1 = x^2/4 - 25 \qquad Q = y^2 \tag{35}$$

then the curve (30) becomes a ring (an ellipse), $x^2/100 + y^2/25 = 1$. Figure 4 shows the corresponding ring soliton structure for the field u (28) with (35) and $a_1 = a_2 = A - 1 = 0$.

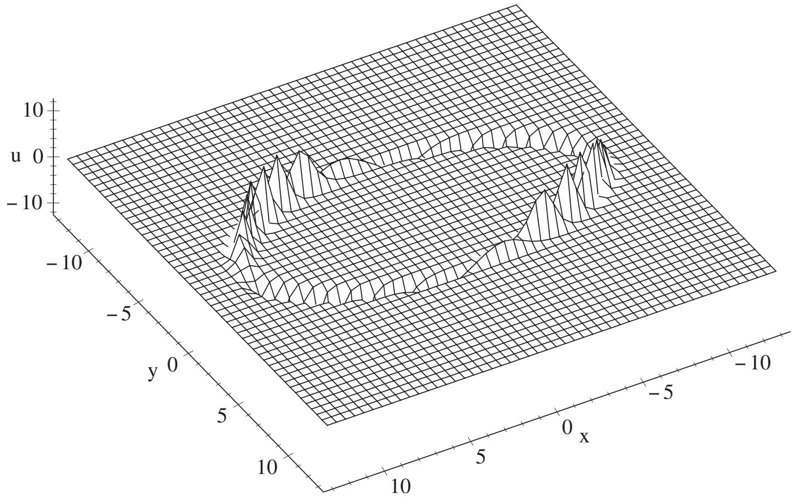


Figure 4. A special ellipse ring soliton solution for the field u (22) with (35) and $a_1 = a_2 = A - 1 = 0$.

3.6. Standing and moving breather-like structures

Obviously, if some types of periodic function of time t are included in the above types of localized solution, then all these types of solution become the corresponding breathers. For instance, if we change P and Q in (35) as

$$P = (\cos t + 9/8)((x - v_1 t)^2/4 - 25) \qquad Q = y^2 + \sin t + 10/9 \tag{36}$$

then the static ring soliton becomes a moving (or standing for $v_1 = 0$) breathe like ellipse ring soliton solution.

From equation (36), we know that the ring breather solution (28) with (36) breathes not only in its amplitude but also in its shape (radii) of the ellipse.

Figure 5 is a plot of a dromion type of breather solution of (22) with $a_1 = A = 10$, $a_2 = 20$ and

$$p = \exp(x(\cos(t) + 20/19)) \qquad q = \exp(y)\exp(\sin(t)) \tag{37}$$

at $t = -\pi, -\pi/2$ and 0. From figures 5(a)–(c), we can see that this type of breather solution breathes also not only in its amplitude but also in its shape. In figure 5(a), we have used a quite different scale of x because the breather has a quite extended shape in the x direction for $t = -\pi$ while the breather has the narrowest shape in the x direction when $t = 0$.

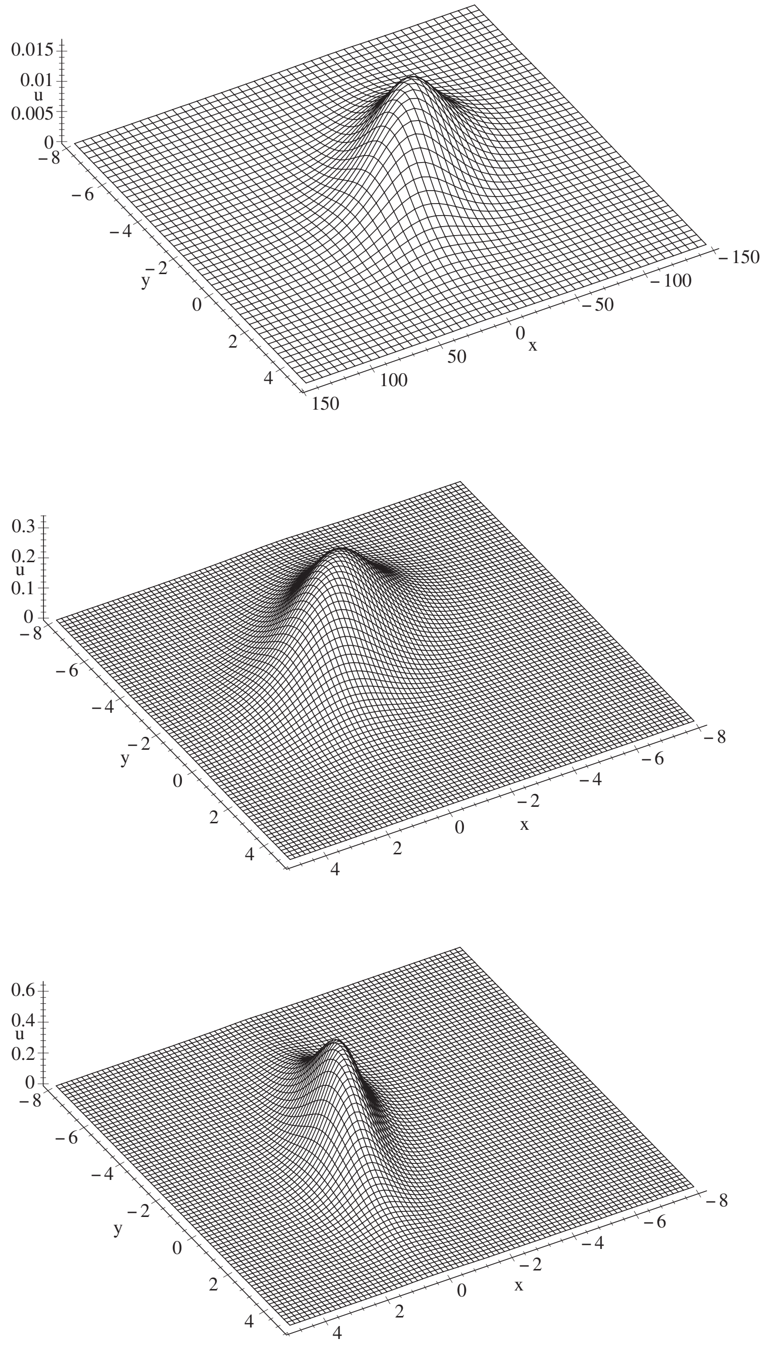
3.7. Instanton-like excitations

If a further decaying factor of time t is contained in p and q of (22), say, multiplying p or q by $\mathrm{sech}(t)$, then the localized structures listed in (i)–(vi) become the instanton-like solutions.

Figure 6 plots another type of instanton solution of the field u shown by (28) with

$$P = x^3 + t^2 - 2 \qquad Q = y^2 + 2t^2 - 4 \qquad a_1 = A = 10 \qquad a_2 = 20 \tag{38}$$

at $t = 0, \pm\sqrt{2 - \ln(20)/8} \equiv t_0$ and ± 2.5. From figures 6(a)–(c), we see that the amplitude(s) of the dromion(s) will decay rapidly as $|t|$ increases.

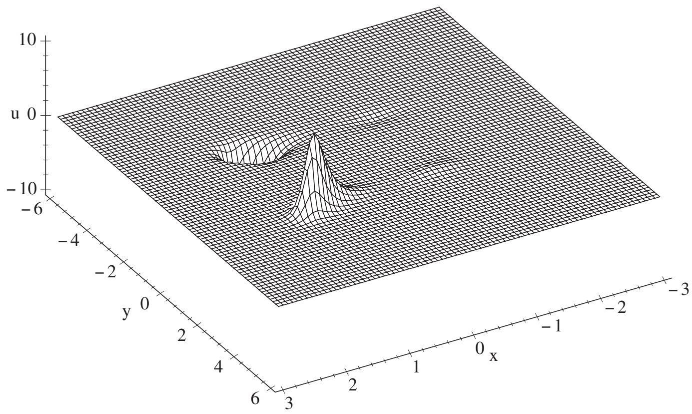


(*a*)

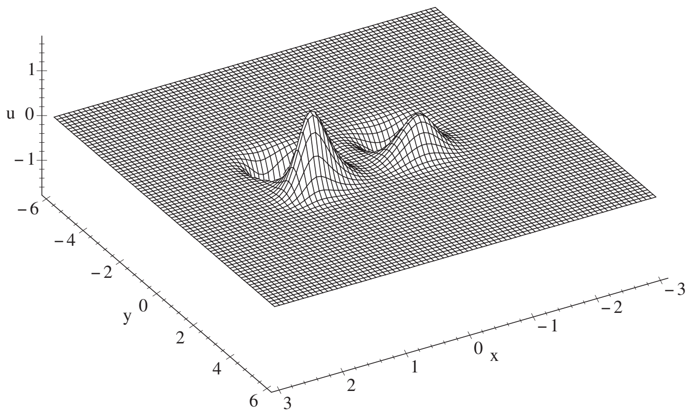
(*b*)

(*c*)

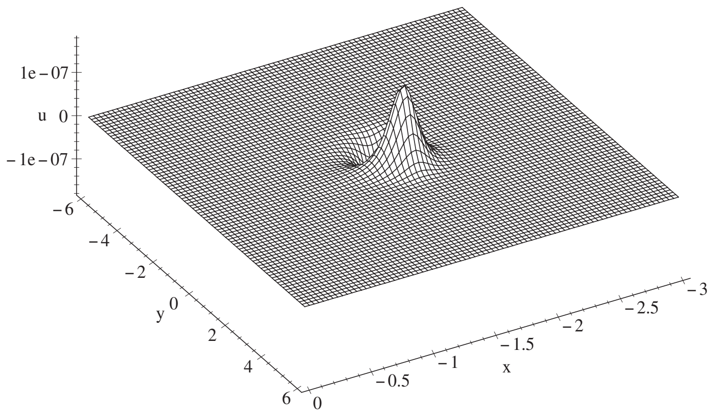
Figure 5. A single-dromion-type breather solution for the field u (22) with (37) and $a_1 = A = 10$, $a_2 = 20$ at times (*a*) $t = -\pi$, (*b*) $t = -\pi/2$ and (*c*) $t = 0$.



(*a*)



(*b*)



(*c*)

Figure 6. An evolution plot of the instanton solution (28) with (38) driven by two separating curved solitons at times (*a*) $t = 0$, (*b*) $t = \pm\sqrt{2 - (\ln 20)/8}$ and (*c*) $t = \pm 2.5$.

The dromion solution shown by figure 6 is driven by two moving curved-line solitons located at

$$x^3 + y^2 + 3t^2 + \tfrac{1}{2}\ln 10 - 6 = 0 \tag{39}$$

$$x^3 - y^2 - t^2 - \tfrac{1}{2}\ln 2 + 2 = 0. \tag{40}$$

For $|t| < t_0$, the curves (39) and (40) have two cross points. So, in principle, (28) with (38) is a two-dromion solution for $|t| < t_0$ as shown in figure 6(a) at $t = 0$. When $|t| = t_0$, two cross points (and then two dromions) are degenerated to one. As $|t|$ increases to over the critical value t_0, there is no cross point of the curves (39) and (40) and only one dromion is located at the closed point of the curves. As the time $|t|$ becomes longer, the distance between the curves will be larger. As the curves become far away each other, the dromion disappears rapidly (the amplitude of the dromion decays exponentially). From figures 6(a)–(c), we can see that as the time $|t|$ increases from 0 to 2.5 the amplitude of the dromion decays tremendously from $\sim$10 to $\sim 10^{-7}$!

4. Summary and discussion

In summary, starting from a Bäcklund transformation and using the variable separation approach for the (2 + 1)-dimensional ANNV equation, we may obtain many new types of multi-soliton solution because of the existence of the arbitrary functions appearing in the seed solution and in the variable separation procedure. By selecting the arbitrary functions appropriately, the multiple localized solution (22) may be dromions, lumps, ring solitons, breathers, instantons etc.

The dromions may be driven by some sets of straight-line solitons and curved-line solitons. Dromions may located not only at the cross points of the curved lines but also the closed points of the curved lines. (2 + 1)-dimensional breathers may breathe not only in amplitude but also in shape. The instantons in (2 + 1) dimensions may have also quite rich structures. Every type of localized structure, such as dromions, lumps, ring solitons and breathers, may become instantons. For instance, as shown in figure 6, the dromion solution driven by two curved lines is an instanton if the curves are separated far apart as the time increases. The richness of the (2 + 1)-dimensional solitons (and solitary waves) may be found also in other high-dimensional models. For instance, using a similar method to the NNV equation, we may obtain also similar abundant localized coherent structures like the ring dromions, breathers and instantons [24]. By means of the standard classical Lie approach to the 2DsG equation proposed by Konopelchenko and Rogers, we pointed out that the 2DsG equation possesses also quite rich localized coherent structures like curved solitons, dromions, ring-type (basin-, plateau- and bowl-like) solitons and instantons [14]. Actually, even for some types of high-dimensional nonintegrable model [25], there may be quite rich localized structures.

The variable separation method in linear physics is a very powerful method. Now we have extended the variable separation approach to some (2 + 1)-dimensional nonlinear integrable models such as the DS, asymmetric DS, ANNV and NNV equations. We hope that the method may also be used for other (2 + 1)-dimensional integrable (and/or even nonintegrable) models. More about the method and the properties of the multiple localized coherent solutions are worthy of further study.

Acknowledgments

This paper was supported by the National Outstanding Youth Foundation of China, the National Natural Science Foundation of China and the National Science Foundation of Zhejiong Province.

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